Convex Combination Based Target Localization with Noisy Angle of Arrival Measurements

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Abstract: This report presents an in-depth exploration of the MUSIC (Multiple Signal Classification) algorithm, a cornerstone in signal processing and its diverse applications. After a comprehensive literature review, four pivotal studies were identified: 'An Improved MUSIC Algorithm Based on Subspace Projection Method', 'A High Resolution Technique for Multidimensional NMR Spectroscopy', 'Improved MUSIC-Based SMOS RFI Source Detection and Geolocation Algorithm', and 'Target Localization Using Convex Combination-Based AoA (Angle of Arrival) Algorithm'. The focus of this investigation narrows on the latter, involving a detailed implementation and analysis using MATLAB. This study not only es elucidates the theoretical underpinnings of the chosen algorithm but also critically examines its practical outcomes. The report is structured to first provide a summarised overview of the four selected papers, followed by a mathematical dissection of the implemented algorithm, culminating in a detailed presentation of the results obtained. This abstract offers a glimpse into the comprehensive journey from theory to application, highlighting the versatility and effectiveness of the MUSIC algorithm in the realm of signal processing.

Keywords: MUSIC algorithm, subspace approximation methods, rank estimation, convex optimization, target localization

I. Introduction

The Multiple Signal Classification (MUSIC) algorithm is a technique used to estimate the direction of arrival (DOA) of multiple signals. This algorithm exploits the eigenstructure of the covariance matrix derived from sensor array data and distinguishes between the signal and noise subspaces based on the fact that they are orthogonal to each other. Data is assumed to be collected by an array of N sensors, each receiving narrowband signals from M distinct sources. These signals are also assumed to arrive from various directions making it difficult to pinpoint their direction of arrival. To do this, we must first estimate the covariance matrix R from the collected sample data matrix X of size $N \times K$. Here, K represents the number of snapshots of the data. The covariance matrix can be constructed by $R = \frac{XX^H}{K}$.

Then, we perform an eigenvalue decomposition of R. The resulting eigenvalues and eigenvectors, sorted in descending order of the magnitude of the eigenvalues, are subsequently segregated into signal and noise subspaces. The eigenvectors associated with the largest M eigenvalues form the signal subspace, while the remaining vectors constitute the noise subspace.

The core functionality of the MUSIC algorithm lies in its spectral calculation. For each direction of interest a steering vector $a(\theta)$ exist st that describes a wave impinging from that particular angle. The MUSIC spectrum is then computed as

$$P(\theta) = \frac{1}{a^H(\theta)E_N E_N^H a(\theta)}$$

where E_N is a matrix containing the eigenvectors of the noise subspace. This spectral formulation allows us to search for peaks where the location of these peak provides the estimated DOA of a particular source signal.

In this paper, we study four different applications of the MUSIC algorithm described above. First, an enhanced version of MUSIC is introduced in which a weighting function is applied to the eigenfunctions of the noise subspace. This method improves the resolution of the algorithm under a low signal-to-noise (SNR) environment or when few snapshots are provided.

Then, we cover an application of the MUSIC algorithm for estimating parameters of a 2D damped sinusoid to be used in determining protein structures. Accurate estimation of the nuclear decay frequencies is required for uncovering structural features, but struggles to cope with drawbacks of traditional numerical methods. This novel application of MUSIC allows us to provide a frequency estimation using multidimensional damped sinusoids uncovered by NMR experiments.

Next, we cover the application of MUSIC for geolocating radio frequency interference (RFI) sources. Data collected by passive satellite systems is compromised by the presence of anthropogenic source signals. To identify and turn off these emitters, we must first be able to correctly uncover the number of sources present in the data, and then subsequently locate the source signal with high geospatial resolution. This work provides a new source signal estimation based on the magnitude of recovered eigenvalues.

Finally, we examine a target localization method that utilizes noisy angle-of-arrival data and redefines this traditional approach as a convex problem. This paper introduces a framework for transforming the estimation of the target's location into a convex format. This transformation aims to yield a unique global solution, even in the presence of non-convex inverse trigonometric components. Additionally, we offer an in-depth exploration of this method, including computational simulations conducted in MATLAB.

II. Overview

In this section, we offer summaries of our highlighted papers before delving into the paper we selected for detailed analysis. Section 3 presents the mathematical analysis, followed by Section 4, which focuses on experimental design. Finally, in Section 5, we provide the results and our interpretation of them.

A. An Improved MUSIC Algorithm Based on Subspace Projection Method

1) Introduction and Problem Statement: The Multiple Signal Classification (MUSIC) algorithm is a widely recognized method in signal processing, particularly for estimating the Direction of Arrival (DOA) of signals. It leverages the eigenstructure of the received signal covariance matrix to distinguish between signal and noise subspaces, enabling accurate DOA estimation. Its applications span various fields, including radar, sonar, and telecommunications. Despite its widespread use, the traditional MUSIC algorithm faces challenges in environments characterized by low Signal-to-Noise Ratio (SNR) and small snapshot numbers. In such scenarios, the algorithm's performance in accurately estimating DOA is significantly reduced. This limitation arises due to the less distinct separation between the signal and noise subspaces under these conditions, leading to decreased resolution and reliability.

2) Improved MUSIC Algorithm: The improved MUSIC algorithm [6] starts with the estimation of the covariance matrix of received signals $\hat{R}$, calculated as $\hat{R} = \frac{1}{L}\sum_{n=1}^{L} x(n)x^H(n)$. Here, $x(n)$ represents the received signals and L is the number of snapshots. Then, the matrix $\hat{R}$ undergoes eigenvalue decomposition: $\hat{R} = \sum_{i=1}^{M} \lambda_i v_i v_i^H$, where λ_i are the eigenvalues, and v_i are the corresponding eigenvectors. This step is crucial for separating the signal and noise subspaces. After, the eigenvalues are then arranged in descending order, and the algorithm separates them into signal and noise subspaces. The signal subspace comprises the eigenvectors corresponding to the largest eigenvalues, while the remaining form the noise subspace. Finally, the novel aspect of the improved algorithm is the eigenvalue correction in the noise subspace: $\tilde{\lambda}_i = \lambda_i + \alpha$. The correction parameter α is introduced to enhance the performance in low SNR and small snapshot scenarios. Consequently, the improved noise subspace $\tilde{U}_N$ is then formed using the corrected eigenvalues $\tilde{\lambda}_i$ and their corresponding eigenvectors, leading to an enhanced resolution of the MUSIC algorithm.

3) Spatial Spectrum Function and DOA Estimation: The improved MUSIC algorithm [6] utilizes a new spatial spectrum function defined as $P(\theta) = \frac{1}{a^H(\theta)\tilde{U}_N\tilde{U}_N^H a(\theta)}$. In this equation, $a(\theta)$ is the steering vector corresponding to the angle of arrival θ, and $\tilde{U}_N$ is the improved noise subspace obtained from the eigenvalue correction method. The DOA is estimated by finding the peaks of the spatial spectrum function $P(\theta)$. The angles corresponding to these peaks indicate the estimated directions of arrival of the signals. This improved spatial spectrum function enhances the algorithm's ability to accurately estimate DOA in environments with low SNR and limited snapshots, addressing the limitations of the traditional MUSIC algorithm.

4) Simulation and Results: The paper [6] conducts simulations to evaluate the performance of the improved MUSIC algorithm. These simulations involve scenarios with varying signal-to-noise ratios (SNR) and different numbers of snapshots to mimic real-world signal conditions. The results demonstrate a significant enhancement in the algorithm's ability to estimate the DOA under low SNR and small snapshot conditions. The improved MUSIC algorithm shows better resolution and accuracy compared to the traditional MUSIC algorithm, validating its effectiveness in challenging signal environments.

5) Discussion: In this section, I would like to discuss in class question regarding what happens when alpha is added to the eigenvalues in terms of noise subspace. When alpha (α) is added to the noise eigenvalues, it indeed modifies the noise subspace. The noise subspace is still spanned by the same eigenvectors. However, the modification lies in the weighting of these vectors. By adjusting the eigenvalues, the algorithm changes how these vectors contribute to the overall noise subspace. This adjustment doesn't change the basis of the subspace but rather the importance assigned to each vector within it. Essentially, what changes is the scale or influence of each vector in the noise subspace. This leads to a more clear distinction between the signal and noise subspaces, especially in challenging conditions like low SNR. This enhances the algorithm's ability to accurately estimate the DOA.

6) Conclusion: The improved MUSIC algorithm [6] demonstrates a significant advancement in DOA estimation, particularly in challenging environments with low SNR and limited snapshots. The introduction of the eigenvalue correction method and the formulation of a new spatial spectrum function are pivotal in enhancing the algorithm's performance.

7) Experimental Design:: In my attempt to reproduce experiments in the paper, I also used a linear antenna array with 8 elements. The spacing between each array element is set to half the wavelength ($\lambda/2$) of the frequency signal 60 Hz. The experiment simulates two non-correlated signal sources at 15° and 60° angles of incidence. Gaussian white noise is added to the signals and setting the Signal-to-Noise Ratio (SNR) to 0 dB, and number of snapshot is 50. As opposed to paper, I should note that I use $\phi_k = 2\pi d cos(\theta_k)/\lambda$ in my steering vector calculation. Then, the received signals are processed using both the traditional and the improved MUSIC algorithms. The focus is on comparing the DOA estimation capabilities under different conditions like by varying parameters such as SNR levels and number of snapshots. From Figure 1 and 2, I observe that improved algorithm has better ability to distinguish between 2 sources in case of low SNR and small snapshots.

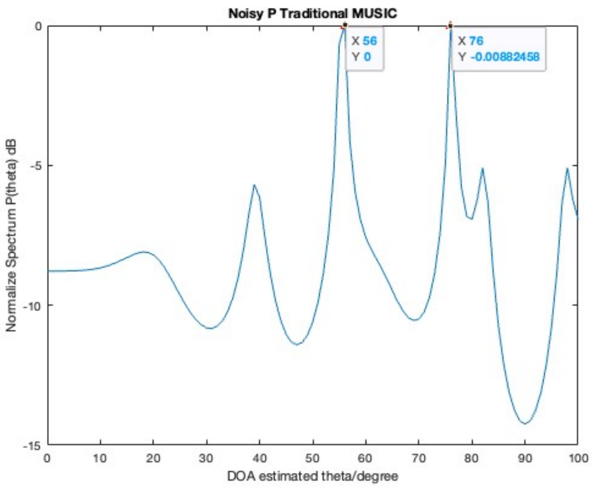

Fig. 1: Spatial Spectrum Function of MUSIC

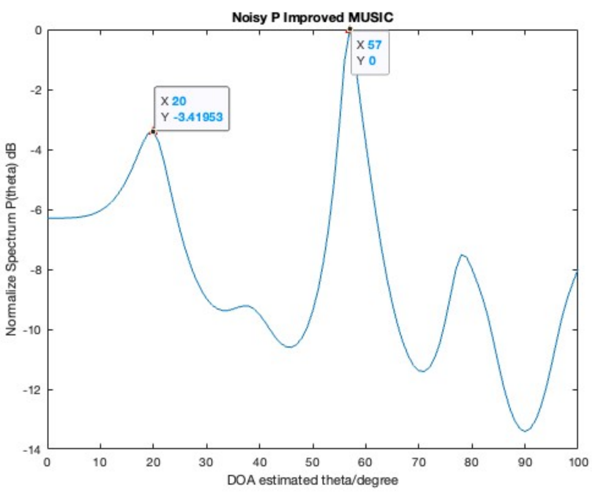

Fig. 2: Spatial Spectrum Function of Improved MUSIC

B. A High Resolution Technique for Multidimensional NMR Spectroscopy

Accurate determination of the 3-dimensional structure of proteins is critical to facilitate understanding of protein structure/function relationships relevant to human health and potential drug targets. Among the techniques employed to elucidate protein structure is nuclear magnetic resonance (NMR) spectroscopy, in which a sample is placed in a strong magnetic field and the decay frequencies of target nuclei are measured through application of carefully tuned radio frequency signals. Through quantum mechanics, the decay frequencies can be used to calculate internuclear distances for all atoms in the protein, thereby yielding a 3D structure. Accurate estimation of the nuclear decay frequencies is essential for high resolution structural determination, and is often plagued by artifacts of the numerical methods used for frequency estimation. Li et al. propose a novel method (D-MUSIC) for frequency estimation based on a manipulation of the MUSIC algorithm to enable calculations with the multidimensional damped sinusoids produced by NMR experiments [1].

The D-MUSIC algorithm extends the original MUSIC algorithm by allowing non-stationary (damped) signals, which requires estimation of an additional parameter (damping coefficient) in addition to the frequency. Further, the algorithm extends the MUSIC algorithm to multidimensional data series.

The D-MUSIC algorithm extends the original MUSIC algorithm by allowing non-stationary (damped) signals, which requires estimation of an additional parameter (damping coefficient) in addition to the frequency. Further, the algorithm extends the MUSIC algorithm to multidimensional data series. Rather than using singular value decomposition of the correlation matrix to define the signal and noise subspaces, the D-MUSIC method utilizes a prediction matrix, $\mathbf{A}$, due to the inability to calculate the correlation matrix for non-stationary signals. The prediction matrix is constructed as a Hankel matrix of the first $N/2$ observations of the signal vector $y(t)$. $\mathbf{A}$ can then be written as

$$\mathbf{A} = \begin{pmatrix} y(0) & y(1) & \dots & y(\frac{N}{2}-1) \\ y(1) & y(2) & \dots & y(\frac{N}{2}) \\ \vdots & \vdots & \ddots & \vdots \\ y(\frac{N}{2}-1) & y(\frac{N}{2}) & \dots & y(N-1) \end{pmatrix} \tag{1}$$

$$\mathbf{A} = \sum_{k=1}^{K} c_k \mathbf{r}_l(s_k)\mathbf{r}_r^T(s_k) + \mathbf{W} = \mathbf{S}_l \mathbf{C} \mathbf{S}_r^T + \mathbf{W} \tag{2}$$

where each column of the right signal matrices $\mathbf{S}_r$ are right signal vectors $(\mathbf{r}_r(s_k))$ consisting of columns of a Vandermonde transpose, c_k is a constant weighting factor, $\mathbf{W}$ is a square matrix of white noise and K is the model order. Singular value decomposition of the prediction matrix $\mathbf{A}$ then yields an orthonormal basis spanning the signal and noise subspaces similar to how the correlation matrix is used in the standard MUSIC algorithm. Computing the inverse of the correlation matrix of the noise vectors left ant right multiplied by each signal vector while scanning across frequencies produces peaks at the frequencies most orthogonal to the noise subspace, thereby identifying the unknown frequencies. Damping coefficients are then estimated through methods established for autocorrelation functions of random sequences which are not directly addressed in the paper.

Extension of this method to multidimensional signals follows from an expansion of the prediction matrix. for multidimensional signals, each entry in the prediction matrix is itself a prediction matrix, creating an overall nested Hankel matrix structure. This ensures that the prediction matrix is full rank, which is important for the subsequent decomposition steps.

To improve the convergence of the estimation, the multidimensional algorithm begins with frequency estimation with zero damping (see Table 1). This approach generally results in a convex optimization problem yielding first pass stationary frequencies near the true frequencies of the non-stationary signals. By iteratively estimating the damping coefficients and re-estimating the frequencies, the method achieves rapid convergence.

While the method described appears to resolve the frequencies and damping coefficients with good accuracy under challenging signal to noise conditions, it still has several limitations. In particular, the model requires a priori knowledge of the rank of the matrix. While this may be approximated through knowledge of the amino acid sequence which can be determined via other experimental methods, the paper does not provide any numerical techniques for rank determination. In addition, the method requires an initial estimation of parameters based on a stationary signal approximation to generate an initial guess that will produce a convex estimation problem. When signals decay quickly, as with ^{15}N or ^{57}Fe nuclei, this initialization step could lead to poor convergence. Despite these limitations, Li et al. demonstrates improved results on real data retrieved from the National Institute of Health's multidimensional NMR database relative to current Fourier transform based techniques. By leveraging the orthogonality of the noise subspace, the D-MUSIC method effectively limits the influence of experimental noise and produces a cleaner spectral estimation with fewer artifacts, ultimately resolving into increased accuracy in determining 3D protein structure. Augmenting the technique discussed here by including rank estimation or virtual anchors to improve convergence with quickly decaying signals may improve overall results.

C. Improved MUSIC-Based SMOS RFI Source Detection and Geolocation Algorithm

Satellite-derived data serves as a foundational resource for monitoring the evolution and interactions of Earth's systems. These satellites often employ electromagnetic observations to infer geophysical parameters with high temporal and spatial resolutions. Given the global scope of these observations, noise contamination from passive or unauthorized sources can significantly impair data collection. Precise geolocation of these disruptive sources is crucial for mitigating their impact and ensuring the integrity of the observations. The MUSIC algorithm offers a robust tool for identifying and locating such problematic sources [3] The European Space Agency's Soil Moisture and Ocean Salinity (SMOS) satellite hosts a microwave imaging radiometer that captures the brightness temperature (BT) of the Earth's surface. RFI sources corrupt this data and must be identified, geolocated, and turned off. Park et al. propose a novel method to do this using only the BT data recorded by the radiometer [3]. Of particular interest is their proposed method for rank estimation under the condition of low SNR or few repeated observations. We may first observe that the covariance matrix can be constructed by cross-correlating the outputs of each antenna pair,

$$\mathbf{R} = \langle \vec{y}\vec{y}^H \rangle$$

Then, an eigenvalue decomposition is performed,

$$\mathbf{R} = \sum_{k=1}^{N} \lambda_k \vec{u}_k \vec{u}_k^H$$

where λ_k and u_k are the constituent eigenvalues with corresponding eigenvectors. This allows the covariance matrix to be decomposed into M_s eigenvalues and eigenvectors to form the signal subspace. In this application the signal subspace is actually comprised of the sources of RFI that we hope to locate and turn off. The noise subspace is naturally then made up of and $N - M_s$ eigenvectors and eigenvalues which are orthogonal to the signal subspace. In order to separate these spaces, we must know the number of signal sources M_s. That is, we must have prior knowledge of the rank of M_s if we wish to decompose the signals. In reality, this is not possible. We do not know the number of sources contaminating our image. Instead, we might estimate the number of sources using the Akaike Information Criterion (AIC) or the Minimum Description Length (MDL). The AIC is given by

$$\begin{aligned} AIC(M_s) = -K(N-1)\log\left(\frac{\prod_{n=M_s+1}^{N} \lambda_n^{\frac{1}{N-M_s}}}{\frac{1}{N-M_s}\sum_{n=M_s+1}^{N} \lambda_n}\right) \\ + M_s(2N - M_s) \end{aligned} \tag{3}$$

and the MDL is given by

$$\begin{aligned} MDL(M_s) = -K(N-1)\log\left(\frac{\prod_{n=M_s+1}^{N} \lambda_n^{\frac{1}{N-M_s}}}{\frac{1}{N-M_s}\sum_{n=M_s+1}^{N} \lambda_n}\right) \\ + \frac{1}{2}M_s(2N - M_s)\log(K) \end{aligned} \tag{4}$$

where K are the number of snapshots taken, i.e. the number of independent observations taken of a certain area. Both methods are comprised of an identical data term and slightly differing penalty terms. The minimization of the AIC or MDL corresponds to the rank of the covariance matrix, M_s. However, neither technique is suitable for use with this data because both the AIC and MDL incorrectly estimate the rank when there is low SNR or a small number of samples [4]. This is simply due to the fact that an over fitting to the observed data is more likely to occur when there is only a single snapshot. SMOS produces a single, relatively long integrated snapshot for each data collection which means that $K = 1$. Thus, we must employ an alternative method to estimate the rank. We can observe that because the signal and noise subspaces are mutually orthogonal, they will appear separate in the eigenvalue plot. If we can estimate where the split occurs in the eigenvalues, then we can estimate the rank of the matrix M_s. In fact, the slope of the eigenvalue plot should reflect this partitioning where the noise regime exhibits a smooth, nearly constant change across eigenvalues, while the slope of the signal subspace has a much larger magnitude.

Unfortunately though, the exact quantification of the slope is dependent on the dataset being used. The authors found that in this scenario, the rank can be obtained by solving the minimum k that satisfies $C_r(k) < \kappa_r$ where $C_r(k)$ is the slope of five consecutive points and $\kappa_r = 1$ is a threshold tuning parameter to limit the variance of the slope. Once the rank M_s of the covariance matrix is known, we may proceed to partition the signal and noise subspaces and to employ the spatial pseudospectrum of MUSIC to determine the directions of arrival of the signals. In conjunction with spacecraft pointing data, we can then geolocate the sources of RFI with a spatial resolution that is superior to the classical DFT.

D. Convex Combination Based Target Localization with Noisy Angle of Arrival Measurements

Traditional AoA-based localization faces challenges due to the non-linear nature of inverse trigonometry functions and noisy sensor data, leading to a non-convex optimization problem that is difficult to solve accurately and efficiently. To overcome these issues, the authors propose a convex combination scheme. This method involves using a highly accurate linear approximation of the inverse trigonometric function, converting the objective into a convex function. This approach allows for efficient solving with a linear least-squares method. The key idea is to express the target's coordinates as a convex combination of a set of virtual anchors placed around its real position. The paper demonstrates through simulations that this method closely approximates the Cramer-Rao lower bound for accuracy and significantly outperforms existing methods in terms of both speed and accuracy.

In their approach, known as Convex-Combination based AoA-Localization [5] (CCAL), the authors adopt a novel approximation scheme to address the difficulties caused by the non-linear inverse trigonometry function. The approximation is highly accurate due to the linear combination forming a convex combination, with coefficients that are non-negative and sum to one. The target coordinates are expressed as a convex combination of virtual anchors, forming a convex hull around the target's position.

III. Mathematical Analysis

In the field of target localization, the main challenge is to precisely determine the target's position, represented as $\mathbf{x}^*$, amidst the noisy Angle of Arrival (AoA) measurements, θ_i, acquired from n physical anchors. The goal is to minimize the difference between the estimated angles and the actual measurements.

The mathematical representation of this problem is as follows:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \sum_{i=1}^{n} \|\theta_i - \phi(\mathbf{x}, \mathbf{a}_i)\|^2 \tag{5}$$

Here:
- n signifies the number of physical anchors,
- $\mathbf{a}_i$ denotes the coordinates of the i^{th} anchor,
- $\theta_i = \tilde{\theta}_i + \tau_i$ represents the noisy measurement, where $\tilde{\theta}_i$ is the actual angle and τ_i is the zero-mean Gaussian noise with variance τ^2,
- τ indicates the noise factor,
- $\phi(p, q)$ is the function calculating the angle between two points p and q, involving a non-linear inverse trigonometric function.

Given that this objective is non-convex, finding a globally optimal solution is challenging. To address this, we introduce m virtual anchors $V = \begin{bmatrix} v_1, & v_2, & \dots, & v_m \end{bmatrix}$ with corresponding nonzero weights $W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$, where $\sum_{i=1}^{m} w_i = 1$.

We then define $x = VW$ as our optimization variable and restructure our problem as:

$$W^* = \arg\min_{W} \sum_{i=1}^{n} \|\theta_i - \phi(VW, a_i)\|^2 \tag{6}$$

$$\text{subject to:} \quad w_i \geq 0, \quad \sum_{i=1}^{m} w_i = 1$$

This can be further expressed as:

$$W^* = \arg\min_{W} \sum_{i=1}^{n} \|\Phi W - \Theta\|^2 \tag{7}$$

$$\text{subject to:} \quad w_i \geq 0, \quad \sum_{i=1}^{m} w_i = 1$$

Where:
- $\Phi = \{\phi_{ij}\} \in \mathbb{R}^{n \times m}, m \geq k+1$ for a k-dimensional problem,
- $\phi_{ij} = \phi(v_j, a_i)$,
- $\Theta = \begin{bmatrix} \theta_1 & \theta_2 & \dots & \theta_n \end{bmatrix}^T$.

After computing W^*, the optimal position $x^* = VW^*$ can be determined.

To conclude, we introduce a method to generate virtual anchors that form a convex hull encompassing the target location. Each physical anchor defines a subregion ω_i, where the target likely resides, as:

$$\omega_i = \{x \mid |\phi(x, a_i) - \theta_i| \leq \alpha\tau\} \tag{8}$$

Here, α is a parameter determining two rays with angles $\theta_i \pm \alpha\tau$, influencing the extent of region ω_i. The feasible region is the intersection of all ω_i, $\Omega = \bigcap_{i=1}^{n} \omega_i$.

Finally, the Root Mean Square Deviation (RMSD) is used to assess the results:

$$\text{RMSD} = \sqrt{\frac{1}{N} \sum_{s=1}^{N} (\Delta d_s)^2}, \quad \Delta d_s = \|\hat{x_s} - x_s\| \tag{9}$$

IV. Experimental Design

To validate the proposed localization approach, we designed an experimental framework using MATLAB. The experiment involves simulating a fixed target within different scenarios, utilizing varying numbers of physical anchors (3, 6, 9, 12, 15) randomly located within a specified detection area. The primary objective is to observe the impact of the number of anchors on the localization accuracy, quantified by the Root Mean Square Deviation (RMSD).

A. *Simulation Parameters:*

- **Anchor Placement:** Physical anchors are randomly distributed within the detection range in each trial.
- **Target Scenarios:** A fixed target position is simulated under varying conditions.
- **Noise Factor Variation:** Different levels of Gaussian noise are added to the AoA measurements.
- **Trial Repetition:** Each scenario is repeated over 1000 trials to ensure statistical significance.

V. Results

A. *Convex Hyperplane Visualization:*

In a representative scenario with 5 physical anchors, the generated convex hyperplane (Figure 1) effectively encompassed the target area. This visualization demonstrates the practical application of virtual anchors in forming a solution space that contains the estimated target location.

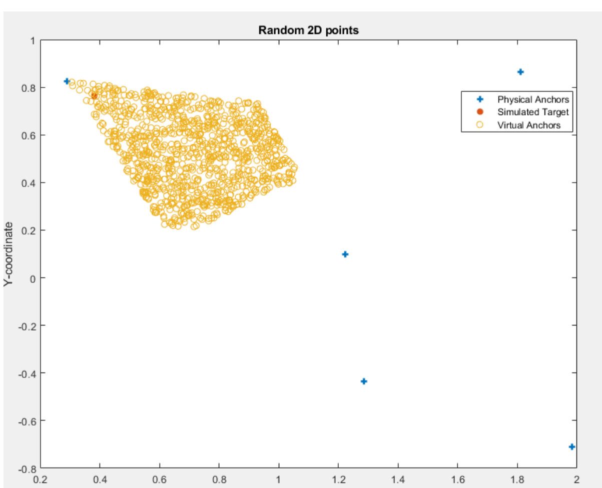

Fig. 3: Convex Hyperplane Visualization.

B. *RMSD Analysis:*

Across the different scenarios with varying numbers of physical anchors, the RMSD values showed a general trend of decrease with an increasing number of anchors. This suggests a direct correlation between the density of physical anchors and the precision of target localization.

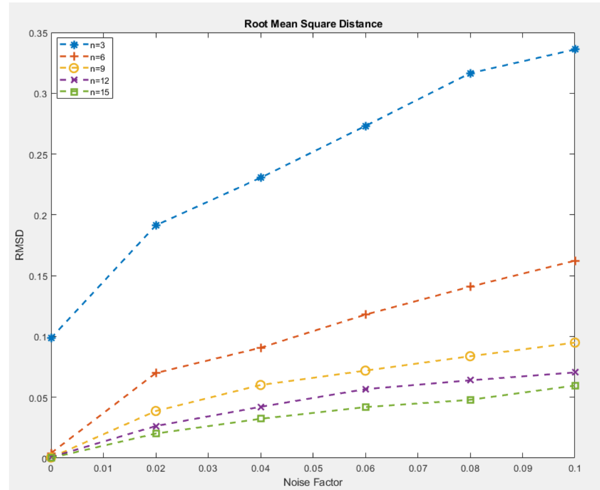

Fig. 4: RMSD vs. Noise Factor Graph.

C. *Impact of Noise Factor:*

The analysis of RMSD against varying noise factors (Figure 2) revealed a linear increase in RMSD with higher noise levels. This outcome underscores the sensitivity of AoA-based localization to measurement noise.

D. *Interpretation and Conclusion:*

The results affirm the viability of the proposed method in accurately localizing a target, especially in scenarios with a higher number of anchors. The efficacy of the virtual anchor system in dealing with non-convex optimization challenges is clearly demonstrated. However, the system's sensitivity to noise suggests a need for robust noise mitigation techniques in practical applications.

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